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Algebraic Generalization of the Ginsparg-Wilson Relation

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Abstract

The generalization of the Ginsparg-Wilson relation to the form $\gamma_5(\gamma_5 D)+(\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2}$ is discussed, where k stands for a non-negative integer and $k = 0$ corresponds to the ordinary Ginsparg-Wilson relation. It is shown that the instanton-related index of all these operators is identical, but the degree of chiral symmetry breaking is different. We thus have an infinite tower of lattice Dirac operators which satisfy the index theorem, but a large enough lattice is required to accomodate such a Dirac operator with a large value of k. We illustrate explicitly a generalization of Neuberger's overlap Dirac operator to the case $k = 1$ and show that the chiral symmetry of the Dirac operator is improved at the near continuum configurations compared to the original overlap Dirac operator. We also briefly sketch the construction of the lattice Dirac operator for any value of k.

1 Introduction

Recent developments in the treatment of fermions in lattice gauge theory are based on a hermitian lattice Dirac operator $\gamma_5 D$ which satisfies the Ginsparg-Wilson relation[1]

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D \tag{1.1}$$

where the lattice spacing a is utilized to make a dimensional consideration transparent, and γ_5 is a hermitian chiral Dirac matrix. An explicit example of the operator satisfying (1.1) and free of species doubling has been given by Neuberger[2]. The relation (1.1) led to an interesting analysis of the notion of index in lattice gauge theory[3]. This index theorem in turn led to a new form of chiral symmetry, and the chiral anomaly is obtained as a non-trivial Jacobian factor under this modified chiral transformation[4]. This chiral Jacobian is regarded as a lattice generalization of the continuum path integral[5]. The very detailed analyses of the lattice chiral Jacobian have been performed[6]-[8]. It is also possible to formulate the lattice index theorem in a manner[9] analogous to the continuum index theorem[10][11]. An interesting chirality sum rule, which relates the number of zero modes to that of the heaviest states, has also been noticed[12].

In this paper we discuss the generalization of the Ginsparg- Wilson relation, which is characterized by a non-negative integer k. It is shown that there exist an infinite tower of lattice Dirac operators which satisfy the index theorem, but a large enough lattice is required to accomodate a Dirac operator with a large value of k.

2 Generalized algebra and its representation

We discuss a generalization of the algebra (1.1) to the form

$$\gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2} \tag{2.1}$$

where k stands for a non-negative integer and $k = 0$ corresponds to the ordinary Ginsparg-Wilson relation. When one defines

$$H \equiv \gamma_5 a D \tag{2.2}$$

(2.1) is rewritten as

$$\gamma_5 H + H\gamma_5 = 2H^{2k+2} \tag{2.3}$$

or equivalently

$$\Gamma_5 H + \Gamma_5 H = 0 \tag{2.4}$$

where we defined

$$\Gamma_5 \equiv \gamma_5 - H^{2k+1}. \tag{2.5}$$

Note that both of H and Γ_5 are hermitian operators.

We now discuss a general representation of the algebraic relation (2.4) following the analysis in Appendix of Ref.[13].(In Ref.[13], the algebra was normalized as $\gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = a(\gamma_5 D)^2$, but here we use the normalization (2.1) to simplify various expressions.) The relation (2.4) suggests that if

$$H\phi_n = a\lambda_n\phi_n, \quad (\phi_n, \phi_n) = 1 \tag{2.6}$$

with a real eigenvalue $a\lambda_n$ for the hermitian operator H, then

$$H(\Gamma_5\phi_n) = -a\lambda_n(\Gamma_5\phi_n). \tag{2.7}$$

Namely, the eigenvalues λ_n and $-\lambda_n$ are always paired if $\lambda_n \neq 0$ and $(\Gamma_5\phi_n, \Gamma_5\phi_n) \neq 0$. We also note the relation, which is derived by sandwiching the relation (2.3) by ϕ_n,

$$(\phi_n, \gamma_5\phi_n) = (a\lambda_n)^{2k+1} \quad for \quad \lambda_n \neq 0. \tag{2.8}$$

Consequently

$$|(a\lambda_n)^{2k+1}| = |(\phi_n, \gamma_5\phi_n)| \leq ||\phi_n|| ||\gamma_5\phi_n|| = 1. \tag{2.9}$$

Namely, all the possible eigenvalues are bounded by

$$|\lambda_n| \leq \frac{1}{a}. \tag{2.10}$$

We thus evaluate the norm of $\Gamma_5\phi_n$

$$
\begin{aligned}
(\Gamma_5\phi_n, \Gamma_5\phi_n) &= (\phi_n, (\gamma_5 - H)(\gamma_5 - H)\phi_n) \\
&= (\phi_n, (1 - (\gamma_5 H + H\gamma_5) + H^2)\phi_n) \\
&= (\phi_n, (1 - 2H^{2k+2} + H^2)\phi_n) \\
&= [1 - (a\lambda_n)^{2k+2} + (a\lambda_n)^2 - (a\lambda_n)^{2k+2}] \\
&= [1 - (a\lambda_n)^2]\{[1 + (a\lambda_n)^2 + ... + (a\lambda_n)^{2k}] \\
&\quad +[(a\lambda_n)^2 + (a\lambda_n)^4 + ... + (a\lambda_n)^{2k}]\}.
\end{aligned}
\tag{2.11}
$$

By remembering that all the eigenvalues are real, we find that ϕ_n is a "highest" state

$$
\Gamma_5\phi_n = 0 \tag{2.12}
$$

only if

$$
[1 - (a\lambda_n)^2] = (1 - a\lambda_n)(1 + a\lambda_n) = 0 \tag{2.13}
$$

for the Euclidean positive definite inner product $(\phi_n, \phi_n) \equiv \sum_x \phi_n^\dagger(x)\phi_n(x)$.
We thus conclude that the states ϕ_n with $\lambda_n = \pm\frac{1}{a}$ are *not* paired by the operation $\Gamma_5\phi_n$ and

$$
\gamma_5 D\phi_n = \pm\frac{1}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n \tag{2.14}
$$

respectively. These eigenvalues are in fact the maximum or minimum of the possible eigenvalues of H/a due to (2.10).

As for the vanishing eigenvalues $H\phi_n = 0$, we find from (2.4) that $H\gamma_5\phi_n = 0$, namely, $H[(1 \pm \gamma_5)/2]\phi_n = 0$. We thus have

$$
\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \phi_n \quad \textit{or} \quad \gamma_5\phi_n = -\phi_n. \tag{2.15}
$$

To summarize the analyses so far, all the normalizable eigenstates ϕ_n of $\gamma_5 D = H/a$ are categorized into the following 3 classes:
(i) $n_\pm$ ("zero modes"),

$$
\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n, \tag{2.16}
$$

(ii) $N_\pm$ ("highest states"),

$$
\gamma_5 D\phi_n = \pm\frac{1}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n, \quad \textit{respectively}, \tag{2.17}
$$

(iii) "paired states" with $0 < |\lambda_n| < 1/a$,

$$
\gamma_5 D\phi_n = \lambda_n\phi_n, \quad \gamma_5 D(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n). \tag{2.18}
$$

Note that $\Gamma_5(\Gamma_5\phi_n) \propto \phi_n$ for $0 < |\lambda_n| < 1/a$.

We thus obtain the index relation

$$\begin{aligned}
Tr\Gamma_5 &\equiv \sum_n(\phi_n,\Gamma_5\phi_n) \\
&= \sum_{\lambda_n=0}(\phi_n,\Gamma_5\phi_n) + \sum_{0<|\lambda_n|<1/a}(\phi_n,\Gamma_5\phi_n) + \sum_{|\lambda_n|=1/a}(\phi_n,\Gamma_5\phi_n) \\
&= \sum_{\lambda_n=0}(\phi_n,\Gamma_5\phi_n) \\
&= \sum_{\lambda_n=0}(\phi_n,(\gamma_5 - H^{2k+1})\phi_n) \\
&= \sum_{\lambda_n=0}(\phi_n,\gamma_5\phi_n) \\
&= n_+ - n_- = index
\end{aligned} \tag{2.19}$$

where $n_\pm$ stand for the number of normalizable zero modes with $\gamma_5\phi_n = \pm\phi_n$ in the classification (i) above. We here used the fact that $\Gamma_5\phi_n = 0$ for the "highest states" and that ϕ_n and $\Gamma_5\phi_n$ are orthogonal to each other for $0 < |\lambda_n| < 1/a$ since they have eigenvalues with opposite signatures.

On the other hand, the relation $Tr\gamma_5 = 0$, which is expected to be valid in (finite) lattice theory, leads to (by using (2.8))

$$\begin{aligned}
Tr\gamma_5 &= \sum_n(\phi_n,\gamma_5\phi_n) \\
&= \sum_{\lambda_n=0}(\phi_n,\gamma_5\phi_n) + \sum_{\lambda_n\neq 0}(\phi_n,\gamma_5\phi_n) \\
&= n_+ - n_- + \sum_{\lambda_n\neq 0}(a\lambda_n)^{2k+1} = 0.
\end{aligned} \tag{2.20}$$

In the last line of this relation, all the states except for the "highest states" with $\lambda_n = \pm 1/a$ cancel pairwise for $\lambda_n \neq 0$. We thus obtain a chirality sum rule $n_+ - n_- + N_+ - N_- = 0$ or

$$n_+ + N_+ = n_- + N_- \tag{2.21}$$

where $N_\pm$ stand for the number of "highest states" with $\gamma_5\phi_n = \pm\phi_n$ in the classification (ii) above. These relations show that the chirality asymmetry at vanishing eigenvalues is balanced by the chirality asymmetry at the largest eigenvalues with $|\lambda_n| = 1/a$. It was argued in Ref.[13] that $N_\pm$ states are the topological (instanton-related) excitations of the would-be species doublers.

All the $n_\pm$ and $N_\pm$ states are the eigenstates of D, $D\phi_n = 0$ and $D\phi_n = (1/a)\phi_n$, respectively. If one denotes the number of states in the classification (iii) above by $2N_0$, the total number of states (the dimension of the representation) N is given by

$$N = 2(n_+ + N_+ + N_0) \tag{2.22}$$

which is expected to be common to all the algebraic relations in (2.1) and to be a constant independent of background gauge field configurations.

We note that all the states ϕ_n with $0 < |\lambda_n| < 1/a$, which appear pairwise with $\lambda_n = \pm|\lambda_n|$, can be normalized to satisfy the relations

$$\begin{aligned}
\Gamma_5\phi_n &= [1-2(a\lambda_n)^{2k+2}+(a\lambda_n)^2]^{1/2}\phi_{-n}, \\
\gamma_5\phi_n &= (a\lambda_n)^{2k+1}\phi_n+[1-2(a\lambda_n)^{2k+2}+(a\lambda_n)^2]^{1/2}\phi_{-n}.
\end{aligned} \tag{2.23}$$

Here ϕ_{-n} stands for the eigenstate with an eigenvalue opposite to that of ϕ_n. These states ϕ_n cannot be the eigenstates of γ_5 since $|(\phi_n, \gamma_5\phi_n)| = |(a\lambda_n)^{2k+1}| < 1$.

We have thus established that the representation of all the algebraic relations (2.1) has a similar structure. In the next Section, we show that the index $n_+ - n_-$ is identical to all these algebraic relations if the operator $\gamma_5 D$ satisfies suitable conditions.

3 Chiral Jacobian and the index relation

The Euclidean path integral for a fermion is defined by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[\int \bar{\psi}D\psi] \tag{3.1}$$

where

$$\int \bar{\psi}D\psi \equiv \sum_{x,y}\bar{\psi}(x)D(x,y)\psi(y) \tag{3.2}$$

and the summation runs over all the points on the lattice. The relation (2.4) is re-written as

$$\gamma_5\Gamma_5\gamma_5 D + D\Gamma_5 = 0 \tag{3.3}$$

and thus the Euclidean action is invariant under the global "chiral" transformation

$$\begin{aligned}
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) &= \bar{\psi}(x) + i\sum_z \bar{\psi}(z)\epsilon\gamma_5\Gamma_5(z,x)\gamma_5 \\
\psi(y) \rightarrow \psi'(y) &= \psi(y) + i\sum_w \epsilon\Gamma_5(y,w)\psi(w)
\end{aligned} \tag{3.4}$$

with an infinitesimal constant parameter ϵ. Under this transformation, one obtains a Jacobian factor

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = J\mathcal{D}\bar{\psi}\mathcal{D}\psi \tag{3.5}$$

with

$$J = \exp[-2iTr\epsilon\Gamma_5] = \exp[-2i\epsilon(n_+ - n_-)] \tag{3.6}$$

where we used the index relation (2.19).

We now relate this index appearing in the Jacobian to the Pontryagin index of the gauge field in a smooth continuum limit by following the procedure in Ref.[9]. We start with

$$Tr\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} = Tr\{\Gamma_5 f(\frac{(H/a)^2}{M^2})\} = n_+ - n_- \tag{3.7}$$

Namely, the index is not modified by any regulator $f(x)$ with $f(0) = 1$ and $f(x)$ rapidly going to zero for $x \rightarrow \infty$, as can be confirmed by using (2.19). This means that you can use *any* suitable $f(x)$ in the evaluation of the index by taking advantage of this property. We then consider a local version of the index

$$tr\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}(x,x) = tr\{(\gamma_5 - H^{2k+1}) f(\frac{(\gamma_5 D)^2}{M^2})\}(x,x) \tag{3.8}$$

where trace stands for Dirac and Yang-Mills indices; Tr in (3.7) includes a sum over the lattice points x. A local version of the index is not sensitive to the precise boundary condition , and one may take an infinite volume limit of the lattice in the above expression.

We now examine the continuum limit $a \to 0$ of the above local expression (3.8)[1]. We first observe that the term

$$tr\{H^{2k+1} f(\frac{(\gamma_5 D)^2}{M^2})\} \tag{3.9}$$

goes to zero in this limit. The large eigenvalues of $H = a\gamma_5 D$ are truncated at the value $\sim aM$ by the regulator $f(x)$ which rapidly goes to zero for large x. In other words, the global index of the operator $TrH^{2k+1}f(\frac{(\gamma_5 D)^2}{M^2}) \sim O(aM)^{2k+1}$.

We thus examine the small a limit of

$$tr\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}. \tag{3.10}$$

The operator appearing in this expression is well regularized by the function $f(x)$, and we evaluate the above trace by using the plane wave basis to extract an explicit gauge field dependence. We consider a square lattice where the momentum is defined in the Brillouin zone

$$-\frac{\pi}{2a} \le k_\mu < \frac{3\pi}{2a}. \tag{3.11}$$

We assume that the operator D is free of species doubling; in other words, the operator D blows up rapidly ($\sim \frac{1}{a}$) for small a in the momentum region corresponding to species doublers. The contributions of doublers are eliminated by the regulator $f(x)$ in the above expression, since

$$tr\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} \sim (\frac{1}{a})^4 f(\frac{1}{(aM)^2}) \to 0 \tag{3.12}$$

for $a \to 0$ if one chooses $f(x) = e^{-x}$, for example.

We thus examine the above trace in the momentum range of the physical species

$$-\frac{\pi}{2a} \le k_\mu < \frac{\pi}{2a}. \tag{3.13}$$

We obtain the limiting $a \to 0$ expression

$$\begin{aligned} &\lim_{a\to 0} tr\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}(x,x) \\ = &\lim_{a\to 0} tr \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(\gamma_5 D)^2}{M^2}) e^{ikx} \end{aligned}$$

[1]This continuum limit corresponds to the so-called "naive" continuum limit in the context of lattice gauge theory.

$$
\begin{aligned}
&= \lim_{L\to\infty}\lim_{a\to 0} tr\int_{-L}^{L}\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})e^{ikx} \\
&= \lim_{L\to\infty} tr\int_{-L}^{L}\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5 f(\frac{(i\gamma_5 \not{D})^2}{M^2})e^{ikx} \\
&\equiv tr\{\gamma_5 f(\frac{\not{D}^2}{M^2})\}
\end{aligned}
\tag{3.14}
$$

where we first take the limit $a\to 0$ with fixed k_μ in $-L\le k_\mu\le L$, and then take the limit $L\to\infty$. This procedure is justified if the integral is well convergent [2]. We also *assumed* that the operator D satisfies the following relation in the limit $a\to 0$

$$
\begin{aligned}
De^{ikx}h(x) &\to e^{ikx}(-\not{k} + i\not{\partial} - g\not{A})h(x) \\
&= i(\not{\partial} + ig\not{A})(e^{ikx}h(x)) \equiv i\not{D}(e^{ikx}h(x))
\end{aligned}
\tag{3.15}
$$

for any *fixed* k_μ, $(-\frac{\pi}{2a} < k_\mu < \frac{\pi}{2a})$, and a sufficiently smooth function $h(x)$. The function $h(x)$ corresponds to the gauge potential in our case, which in turn means that the gauge potential $A_\mu(x)$ is assumed to vary very little over the distances of the elementary lattice spacing.

Our final expression (3.14) in the limit $M\to\infty$ reproduces the Pontryagin number in the continuum formulation

$$
\begin{aligned}
&\lim_{M\to\infty} tr\{\gamma_5 f(\not{D}^2/M^2)\} \\
&= \lim_{M\to\infty} tr\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\gamma_5 f(\not{D}^2/M^2)e^{ikx} \\
&= \lim_{M\to\infty} tr\int\frac{d^4k}{(2\pi)^4}\gamma_5 f\{(ik_\mu + D_\mu)^2/M^2 + \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}/M^2\} \\
&= \lim_{M\to\infty} trM^4\int\frac{d^4k}{(2\pi)^4}\gamma_5 f\{(ik_\mu + D_\mu/M)^2 + \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}/M^2\}
\end{aligned}
\tag{3.16}
$$

where the remaining trace stands for Dirac and Yang-Mills indices. We also used the relation

$$
\not{D}^2 = D_\mu D^\mu + \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}
\tag{3.17}
$$

[2] To be precise, we deal with an integral of the structure $\int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}}dxf_a(x) = \int_{L}^{\frac{\pi}{2a}}dxf_a(x) + \int_{-L}^{L}dxf_a(x) + \int_{-\frac{\pi}{2a}}^{-L}dxf_a(x)$ where $f_a(x)$ depends on the parameter a. (A generalization to a 4-dimensional integral is straightforward.) We thus have to prove that both of $\lim_{a\to 0}\int_{L}^{\frac{\pi}{2a}}dxf_a(x)$ and $\lim_{a\to 0}\int_{-\frac{\pi}{2a}}^{-L}dxf_a(x)$ can be made arbitrarily small if one lets L to be large. A typical integral we encounter in lattice theory has a generic structure $\lim_{a\to 0}\int_{-\pi/2a}^{\pi/2a}dxe^{-[\sin^2 ax+(1-\cos 2ax)^2]/(a^2M^2)} = \lim_{a\to 0}\int_{\epsilon\pi/2a}^{\pi/2a}dxe^{-[\sin^2 ax+(1-\cos 2ax)^2]/(a^2M^2)} + \lim_{a\to 0}\int_{-\pi/2a}^{-\epsilon\pi/2a}dxe^{-[\sin^2 ax+(1-\cos 2ax)^2]/(a^2M^2)} + \lim_{a\to 0}\int_{-\epsilon\pi/2a}^{\epsilon\pi/2a}dxe^{-[\sin^2 ax+(1-\cos 2ax)^2]/(a^2M^2)} = \lim_{a\to 0}\int_{-\epsilon\pi/2a}^{\epsilon\pi/2a}dxe^{-[\sin^2 ax+(1-\cos 2ax)^2]/(a^2M^2)} = \lim_{L\to\infty}\int_{-L}^{L}dxe^{-x^2/M^2}$ and satisfies the above criterion, if one chooses the regulator $f(x)=e^{-x}$: Here ϵ is an arbitrary small fixed parameter, and the left-hand side of this relation stands for a conventional lattice calculation and the right-hand side stands for a continuum calculation.

and the rescaling of the variable $k_\mu \to Mk_\mu$.

By noting $tr\gamma_5 = tr\gamma_5[\gamma^\mu, \gamma^\nu] = 0$, the above expression (after expansion in powers of $1/M$) is written as (with $\epsilon^{1234} = 1$)

$$\begin{aligned}\lim_{M\to\infty} tr\gamma_5 f(\not{D}^2/M^2) &= tr\gamma_5 \frac{1}{2!}\{\frac{ig}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^2 \int \frac{d^4k}{(2\pi)^4} f''(-k_\mu k^\mu) \\ &= \frac{g^2}{32\pi^2} tr\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \end{aligned} \tag{3.18}$$

where we used

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4} f''(-k_\mu k^\mu) &= \frac{1}{16\pi^2}\int_0^\infty f''(x)xdx \\ &= \frac{1}{16\pi^2} \end{aligned} \tag{3.19}$$

with $x = -k_\mu k^\mu > 0$ in our metric.

When one combines (3.7) and (3.18), one reproduces the Atiyah-Singer index theorem (in continuum R^4 space)[10][11]. We note that a local version of the index (anomaly) is valid for Abelian theory also. The global index (3.7) as well as a local version of the index (3.8) are both independent of the regulator $f(x)$ provided [5]

$$f(0) = 1, \quad f(\infty) = 0, \quad f'(x)x|_{x=0} = f'(x)x|_{x=\infty} = 0. \tag{3.20}$$

We have thus established that the lattice index in (3.7) for any algebraic relation in (2.1) is related to the Pontryagin index in a smooth continuum limit as

$$n_+ - n_- = \int d^4x \frac{g^2}{32\pi^2} tr\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{3.21}$$

by assuming the quite general properties of the basic operator D only: The basic relation (2.1) with hermitian $\gamma_5 D$ and the continuum limit property (3.15) *without* species doubling in the limit $a \to 0$. This shows that the instanton-related topological property is identical for all the algebraic relations in (2.1), and the Jacobian factor (3.6) in fact contains the correct chiral anomaly. (We are implicitly assuming that the index (3.7) does not change in the process of taking a continuum limit.)

4 Explicit example of the lattice Dirac operator with k=1

We now discuss an explicit construction of the lattice Dirac operator which satisfies the generalized algebraic relation (2.1) with $k = 1$, though a generalization to an arbitrary k is straightforward as is briefly described in Section 5 later. For this purpose, we first briefly review the construction of the Neuberger's overlap Dirac operator for the ordinary Ginsparg-Wilson relation.

We start with the conventional Wilson fermion operator D_W defined by

$$\begin{aligned}
D_W(x,y) &\equiv i\gamma^\mu C_\mu(x,y) + B(x,y) - \frac{1}{a}m_0\delta_{x,y}, \\
C_\mu(x,y) &= \frac{1}{2a}[\delta_{x^\mu+\hat{\mu}a,y^\mu}U_\mu(y) - \delta_{x^\mu,y^\mu+\hat{\mu}a}U_\mu^\dagger(x)], \\
B(x,y) &= \frac{r}{2a}\sum_\mu[2\delta_{x,y} - \delta_{y^\mu+\hat{\mu}a,x^\mu}U_\mu^\dagger(x) - \delta_{y^\mu,x^\mu+\hat{\mu}a}U_\mu(y)], \\
U_\mu(y) &= \exp[iagA_\mu(y)],
\end{aligned} \tag{4.1}$$

where we added a constant mass term to D_W for later convenience. The parameter r stands for the Wilson parameter. Our matrix convention is that γ^μ are anti-hermitian, $(\gamma^\mu)^\dagger = -\gamma^\mu$, and thus $\not{C} \equiv \gamma^\mu C_\mu(n,m)$ is hermitian

$$\not{C}^\dagger = \not{C}. \tag{4.2}$$

The operator D introduced by Neuberger[2], which satisfies the conventional Ginsparg-Wilson relation (1.1), has an explicit expression

$$aD = \frac{1}{2}[1 + \gamma_5\frac{H_W}{\sqrt{H_W^2}}] = \frac{1}{2}[1 + D_W\frac{1}{\sqrt{D_W^\dagger D_W}}] \tag{4.3}$$

where $D_W = \gamma_5 H_W$ is the Wilson operator defined above, and H_W is hermitian $H_W^\dagger = H_W$.

The physical meaning of this construction becomes more transparent if one considers (naive) near continuum configurations specified by a small a limit with the parameters r/a and m_0/a kept finite. We can then approximate the operator D_W by[14]

$$D_W \simeq i\not{D} + M_n \tag{4.4}$$

for each species doubler, where the mass parameters M_n stand for $M_0 = -\frac{m_0}{a}$ and one of

$$\begin{array}{llll}
\frac{2r}{a} - \frac{m_0}{a}, & (4,-1); & \frac{4r}{a} - \frac{m_0}{a}, & (6,1) \\
\frac{6r}{a} - \frac{m_0}{a}, & (4,-1); & \frac{8r}{a} - \frac{m_0}{a}, & (1,1)
\end{array} \tag{4.5}$$

for $n = 1 \sim 15$; we denoted (multiplicity, chiral charge) in the bracket for species doublers. Here we used the relation valid in the near continuum configurations for the physical species, for example,

$$\begin{aligned}
D_W(k) &= \sum_\mu \gamma^\mu\frac{\sin ak_\mu}{a} + \frac{r}{a}\sum_\mu(1-\cos ak_\mu) - \frac{m_0}{a} \\
&\simeq \gamma^\mu k_\mu - \frac{m_0}{a}
\end{aligned} \tag{4.6}$$

in the momentum representation with vanishing gauge field.

In a symbolic notation, one can then write the overlap Dirac operator as

$$
\begin{aligned}
aD &\simeq \sum_{n=0}^{15}\frac{1}{2}[1+(i\not{D}+M_n)\frac{1}{\sqrt{\not{D}^2+M_n^2}}]|n\rangle\langle n|,\\
a\gamma_5 D &\simeq \sum_{n=0}^{15}(-1)^n\gamma_5\frac{1}{2}[1+(i\not{D}+M_n)\frac{1}{\sqrt{\not{D}^2+M_n^2}}]|n\rangle\langle n|.
\end{aligned}
\tag{4.7}
$$

Here we explicitly write the projection $|n\rangle\langle n|$ for each species doubler. If one chooses the mass parameters so that

$$M_0 = -\frac{m_0}{a} < 0, \quad M_n > 0 \quad for \quad n \neq 0 \tag{4.8}$$

namely

$$0 < m_0 < 2r \tag{4.9}$$

and if one lets all the mass parameters $|M_n|$ become large, one obtains

$$
\begin{aligned}
a\gamma_5 D &\simeq \gamma_5\frac{1}{2}[\frac{i\not{D}}{|M_0|}+\frac{1}{2}\frac{\not{D}^2}{M_0^2}] \quad for \quad n=0,\\
a\gamma_5 D &\simeq (-1)^n\gamma_5\frac{1}{2}[2+\frac{i\not{D}}{M_n}-\frac{1}{2}\frac{\not{D}^2}{M_n^2}] \quad for \quad n\neq 0.
\end{aligned}
\tag{4.10}
$$

If one chooses m_0 to satisfy

$$2a|M_0| = 2m_0 = 1 \tag{4.11}$$

one recovers the correctly normalized continuum Dirac operator for the physical species and $\gamma_5 D \simeq (-1)^n\gamma_5\frac{1}{a}$ for unphysical species doublers. In particular, the first relation in (4.10) can then be written as

$$H \equiv a\gamma_5 D \simeq \gamma_5 a i\not{D} + \gamma_5(\gamma_5 a i\not{D})^2 \tag{4.12}$$

which ensures the conventional Ginsparg-Wilson relation in the leading order. These properties become important in the following discussion.

4.1 Generalized algebra with $k = 1$

We now come back to the generalized algebra (2.1) with $k = 1$

$$H\gamma_5 + \gamma_5 H = 2H^4 \tag{4.13}$$

where $H = a\gamma_5 D$ and $\Gamma_5 = \gamma_5 - H^3$. This algebraic relation implies that

$$\gamma_5 H^2 = [\gamma_5 H + H\gamma_5]H - H[\gamma_5 H + H\gamma_5] + H^2\gamma_5 = H^2\gamma_5 \tag{4.14}$$

Namely, the algebraic relation (4.13) is equivalent to the two relations

$$
\begin{aligned}
&H^3\gamma_5 + \gamma_5 H^3 = 2H^6,\\
&\gamma_5 H^2 - H^2\gamma_5 = 0.
\end{aligned}
\tag{4.15}
$$

If one defines $H_{(3)} \equiv H^3$, the first relation of (4.15) becomes

$$H_{(3)}\gamma_5 + \gamma_5 H_{(3)} = 2H_{(3)}^2 \tag{4.16}$$

with $\Gamma_5 = \gamma_5 - H_{(3)}$, which is identical to the conventional Ginsparg-Wilson relation (1.1). We utilize this property to construct a solution to (4.15). Note that the operator Γ_5 is identical in these three ways of writing in (4.13), (4.15), and (4.16).

The physical condition for the operator H in (4.13) in the near continuum configuration is (Cf.(4.12))

$$H \simeq \gamma_5 ai \not{D} + \gamma_5(\gamma_5 ai \not{D})^4 \tag{4.17}$$

and thus $H_{(3)}$ in (4.16) should satisfy

$$\begin{aligned} H_{(3)} &\simeq [\gamma_5 ai \not{D} + \gamma_5(\gamma_5 ai \not{D})^4]^3 \\ &\simeq (\gamma_5 ai \not{D})^3 + \gamma_5(\gamma_5 ai \not{D})^6 \end{aligned} \tag{4.18}$$

as can be confirmed by noting $\gamma_5 \not{D} + \not{D}\gamma_5 = 0$. Here only the leading terms in chiral symmetric and chiral symmetry breaking terms respectively are written.

One can thus construct a solution for $H_{(3)}$ by

$$H_{(3)} = \frac{1}{2}\gamma_5[1 + D_W^{(3)}\frac{1}{\sqrt{(D_W^{(3)})^\dagger D_W^{(3)}}}] \tag{4.19}$$

where we defined $D_W^{(3)}$ by[3]

$$D_W^{(3)} \equiv i(\not{C})^3 + (B)^3 - (\frac{m_0}{a})^3 \tag{4.20}$$

The operators $\not{C}$,B and the parameter m_0/a are the same as in the original Wilson fermion operator (4.1). By rewriting (4.19) as

$$H_{(3)} = \frac{1}{2}\gamma_5[1 + \gamma_5 H_W^{(3)}\frac{1}{\sqrt{H_W^{(3)} H_W^{(3)}}}] \tag{4.21}$$

in terms of the hermitian $H_W^{(3)} \equiv \gamma_5 D_W^{(3)} = (H_W^{(3)})^\dagger$ and comparing it with (4.3), one can confirm that our operator $H_{(3)}$ satisfies the relation (4.16). The condition (4.18) is satisfied by noting

$$D_W^{(3)} \simeq i(\not{D})^3 + (M_n^{(3)})^3 \tag{4.22}$$

in the near continuum configuration, where the mass parameters are given by

$$\begin{aligned} (M_0^{(3)})^3 &\equiv -(\frac{m_0}{a})^3 \\ (M_n^{(3)})^3 &\equiv \{(\frac{2r}{a})^3 - (\frac{m_0}{a})^3, (\frac{4r}{a})^3 - (\frac{m_0}{a})^3, (\frac{6r}{a})^3 - (\frac{m_0}{a})^3, (\frac{8r}{a})^3 - (\frac{m_0}{a})^3\} \\ & \quad for \quad n \neq 0. \end{aligned} \tag{4.23}$$

[3]It is also possible to use $D_W^{(3)} \equiv i(\not{C})^3 + (B - \frac{m_0}{a})^3$, or any suitable (ultra-local) operator which satisfies $\gamma_5 D_W^{(3)} = (\gamma_5 D_W^{(3)})^\dagger$ and (4.22).

Although we have the same condition on the parameters as before

$$0 < m_0 < 2r \tag{4.24}$$

to avoid the species doublers, the value of m_0 itself is now required to satisfy

$$2(m_0)^3 = 1 \tag{4.25}$$

to ensure the properly normalized physical condition (4.18).

4.2 Reconstruction of H from $H_{(3)}$

We now discuss how to reconstruct H, which satisfies (4.13), from $H_{(3)}$ defined above. The basic idea is to take a real cubic root of $H_{(3)}$ as

$$H = (H_{(3)})^{1/3} \tag{4.26}$$

in such a manner that H thus obtained satisfies the second constraint in (4.15). For this purpose, we first recall the essence of the general representation of the algebra (2.1) analyzed in Section 2, which is applicable to (4.16) as well.

If one defines the eigenvalue problem

$$H_{(3)}\phi_n = (a\lambda_n)^3\phi_n, \quad (\phi_n, \phi_n) = 1 \tag{4.27}$$

one can classify the eigenstates into the 3 classes:
(i) $n_\pm$ ("zero modes"),

$$H_{(3)}\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n, \tag{4.28}$$

(ii) $N_\pm$ ("highest states"),

$$H_{(3)}\phi_n = \pm\phi_n, \quad \gamma_5\phi_n = \pm\phi_n, \quad \textit{respectively}, \tag{4.29}$$

(iii) "paired states" with $0 < |(a\lambda_n)^3| < 1$,

$$H_{(3)}\phi_n = (a\lambda_n)^3\phi_n, \quad H_{(3)}(\Gamma_5\phi_n) = -(a\lambda_n)^3(\Gamma_5\phi_n). \tag{4.30}$$

where

$$\Gamma_5 = \gamma_5 - H_{(3)}. \tag{4.31}$$

Note that $\Gamma_5(\Gamma_5\phi_n) \propto \phi_n$ for $0 < |(a\lambda_n)^3| < 1$.

We obtain the index relation

$$\begin{aligned} Tr\Gamma_5 &\equiv \sum_n (\phi_n, \Gamma_5\phi_n) \\ &= \sum_{\lambda_n=0} (\phi_n, \gamma_5\phi_n) \\ &= n_+ - n_- = index \end{aligned} \tag{4.32}$$

where $n_{\pm}$ stand for the number of normalizable zero modes in the classification (i) above.

We also have a chirality sum rule

$$n_{+} + N_{+} = n_{-} + N_{-} \tag{4.33}$$

where $N_{\pm}$ stand for the number of "highest states" in the classification (ii) above.

If one denotes the number of states in the classification (iii) above by $2N_0$, the total number of states (the dimension of the representation) N is given by

$$N = 2(n_{+} + N_{+} + N_0) \tag{4.34}$$

which is expected to be common to all the fermion operators defined on the same lattice.

Also, all the states ϕ_n with $0 < |(a\lambda_n)^3| < 1$, which appear pairwise with $(a\lambda_n)^3 = \pm|(a\lambda_n)^3|$, can be normalized to satisfy the relations

$$\begin{aligned}
\Gamma_5\phi_n &= [1-(a\lambda_n)^6]^{1/2}\phi_{-n}, \\
\gamma_5\phi_n &= (a\lambda_n)^3\phi_n + [1-(a\lambda_n)^6]^{1/2}\phi_{-n},
\end{aligned} \tag{4.35}$$

where ϕ_{-n} stands for the eigenstate with an eigenvalue opposite to that of ϕ_n.

Based on these general results in Section 2, we first observe that the index $n_{+} - n_{-}$ in (4.32) is identical to the index of the expected solution of (4.13), although $H_{(3)}$ satisfies (4.18). This observation is based on the relation

$$n_{+} - n_{-} \equiv \sum_n (\phi_n, \Gamma_5 f((H_{(3)})^2/(aM)^6)\phi_n) \tag{4.36}$$

which is valid for any regulator with $f(0) = 1$. One can perform the same analysis as in (3.7) in Section 3: The basic ingredient is the condition (4.18) for a physical momentum region in the smooth continuum limit and the absence of species doublers. The calculation analogous to (3.14) then gives

$$n_{+} - n_{-} = \lim_{M\to\infty} Tr\gamma_5 f(\frac{\not{D}^6}{M^6}) = \lim_{M\to\infty} Tr\gamma_5 g(\frac{\not{D}^2}{M^2}) \tag{4.37}$$

with $g(x) \equiv f(x^3)$ and $g(0) = 1$. The right-hand side of this relation shows that the present index is identical to the index of the general operator in (2.1), which includes an expected solution of (4.13). Due to the chirality sum rule (4.33), we also obtain the same value of $N_{+} - N_{-}$ as for an expected solution of (4.13).

The agreement of the index of $H_{(3)}$ with the index of the expected solution H of (4.13) suggests that we can define H *operationally* by

$$H\phi_n \equiv a\lambda_n\phi_n \tag{4.38}$$

by using the *same set* of eigenfunctions and (the cubic roots of) eigenvalues

$$\{\phi_n\}, \quad \{a\lambda_n\} \tag{4.39}$$

as for $H_{(3)}$ in (4.27). Note that the operator $\Gamma_5 = \gamma_5 - H_{(3)} = \gamma_5 - H^3$, which reverses the signature of eigenvalues of "paired states" and defines the index, is consistently chosen to be identical for (4.16) and for (4.38)[4].

We can then confirm the second constraint in (4.15) and the defining algebraic relation (4.13) for *any* "paired state" ϕ_n,

$$\begin{aligned}
[H^2\gamma_5 - \gamma_5 H^2]\phi_n &= H^2\gamma_5\phi_n - \gamma_5(a\lambda_n)^2\phi_n \\
&= H^2\{(a\lambda_n)^3\phi_n + [1-(a\lambda_n)^6]^{1/2}\phi_{-n}\} \\
&\quad -(a\lambda_n)^2\{(a\lambda_n)^3\phi_n + [1-(a\lambda_n)^6]^{1/2}\phi_{-n}\} \\
&= 0
\end{aligned} \tag{4.40}$$

and

$$[\Gamma_5 H + H\Gamma_5]\phi_n = \Gamma_5(a\lambda_n)\phi_n - a\lambda_n(\Gamma_5\phi_n) = 0 \tag{4.41}$$

where we used the relations in (4.35) and the definition (4.38). For "zero modes" and the "highest states", which are the eigenstates of γ_5, the condition $[H^2\gamma_5 - \gamma_5 H^2]\phi_n = 0$ obviously holds, and the relation $[\Gamma_5 H + H\Gamma_5]\phi_n = 0$ is also confirmed.

The general representation of the algebra (4.13) is obtained from the *standard representation*, which is defined by H in (4.38), γ_5 in (4.35), and the state vectors $\{\phi_n\}$ in (4.39), by applying a suitable unitary transformation.

5 Discussion

When one considers the algebraic relation with a constant R

$$\gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = 2Ra^{2k+1}(\gamma_5 D)^{2k+2} \tag{5.1}$$

instead of (2.1), one can eliminate the paprameter R by a scale transformation

$$D \to D' = R^{1/(2k+1)}D. \tag{5.2}$$

The path integral

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[\int \bar{\psi}D'\psi] \tag{5.3}$$

is equivaqlent to

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[\int \bar{\psi}D\psi] \tag{5.4}$$

after absorbing the parameter $R^{1/(2k+1)}$ into $\bar{\psi}$, at least in a well regularized lattice path integral. Consequently, the parameter R and also the factor a^{2k+1} do not have an intrinsic physical significance[5].

[4] This means that a perturbative calculation of the chiral Jacobian (and chiral anomaly) for the theory defined by (4.13) is performed by $Tr\Gamma_5 = Tr(\gamma_5 - H_{(3)})$ in terms of $H_{(3)}$ in (4.19).

[5] However, when one includes a Yukawa interaction, for example, this scaling argument need to be refined.

In contrast, the power of $(\gamma_5 D)^{2k+2}$ in the right-hand side of (5.1) has an intrinsic physical meaning. One may recall the near continuum expressions (4.12) and (4.17)

$$\begin{aligned} H &\simeq \gamma_5 ai \not{D} + \gamma_5(\gamma_5 ai \not{D})^2 \quad for \quad k=0, \\ H &\simeq \gamma_5 ai \not{D} + \gamma_5(\gamma_5 ai \not{D})^4 \quad for \quad k=1 \end{aligned} \tag{5.5}$$

respectively. The first terms in these expressions stand for the leading terms in chiral symmetric terms, and the second terms in these expressions stand for the leading terms in chiral symmetry breaking terms. This shows that one can improve the chiral symmetry by choosing a large parameter k.

The Dirac operator for such a value of k is constructed by rewriting (2.1) as a set of relations

$$\begin{aligned} &H^{2k+1}\gamma_5 + \gamma_5 H^{2k+1} = 2H^{2(2k+1)}, \\ &H^2\gamma_5 - \gamma_5 H^2 = 0, \end{aligned} \tag{5.6}$$

with $H = a\gamma_5 D$. The first of these relations (5.6) becomes identical to the ordinary Ginsparg-Wilson relation if one defines $H_{(2k+1)} \equiv H^{2k+1}$. One can construct a solution to (5.6) by following the prescription in Section 4

$$H_{(2k+1)} = \frac{1}{2}\gamma_5[1 + D_W^{(2k+1)}\frac{1}{\sqrt{(D_W^{(2k+1)})^\dagger D_W^{(2k+1)}}}] \tag{5.7}$$

where

$$D_W^{(2k+1)} \equiv i(\not{C})^{2k+1} + B^{2k+1} - (\frac{m_0}{a})^{2k+1} \tag{5.8}$$

The operator H is then finally defined by (in the representation where $H_{(2k+1)}$ is diagonal)

$$H = (H_{(2k+1)})^{1/2k+1} \tag{5.9}$$

in such a manner that the second relation of (5.6) is satisfied. However, one need to use a large enough lattice to accomodate the operator H with a large k, since the operator (5.8) correlates lattice points far apart from each other for a large k. In practice, it would thus be interesting to see how the chiral properties are improved if one uses the operator with $k = 1$, which has been analyzed in detail in this paper, instead of the conventional overlap Dirac operator with $k = 0$.

As for the chiral fermions on the lattice, our general algebra (2.1) satisfies the decomposition

$$D = \frac{(1+\gamma_5)}{2}D\frac{(1-\hat{\gamma}_5)}{2} + \frac{(1-\gamma_5)}{2}D\frac{(1+\hat{\gamma}_5)}{2} \tag{5.10}$$

with

$$\hat{\gamma}_5 \equiv \gamma_5 - 2a^{2k+1}(\gamma_5 D)^{2k+1}, \qquad (\hat{\gamma}_5)^2 = 1 \tag{5.11}$$

by noting $\gamma_5(\gamma_5 D)^2 = (\gamma_5 D)^2\gamma_5$. This decomposition has the same form as for the overlap operator D satisfying the ordinary Ginsparg-Wilson relation. It is thus expected that one can apply the same considerations as in Refs.[15] and [16] to our general Dirac operator also. In particular, the fermion number non-conservation of the chiral theory defined by

$$
\begin{aligned}
&\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{\int \bar{\psi}D_L\psi\} \\
&\equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{\int \bar{\psi}\frac{(1+\gamma_5)}{2}D\frac{(1-\hat{\gamma}_5)}{2}\psi\}
\end{aligned} \tag{5.12}
$$

follows from the fermion number transformation

$$
\psi \rightarrow e^{i\alpha}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{-i\alpha}. \tag{5.13}
$$

If one remembers that the functional spaces of the variables ψ and $\bar{\psi}$ are specified by the projection operators $(1-\hat{\gamma}_5)/2$ and $(1+\gamma_5)/2$, respectively, the Jacobian factor for the transformation (5.13) is given by[15]

$$
\begin{aligned}
J &= \exp\{i\alpha Tr[\frac{(1+\gamma_5)}{2} - \frac{(1-\hat{\gamma}_5)}{2}]\} \\
&= \exp\{i\alpha Tr[\gamma_5 - (\gamma_5 aD)^{2k+1}]\} = \exp\{i\alpha[n_+ - n_-]\}
\end{aligned} \tag{5.14}
$$

where the index is defined in (2.19).

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